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AUTHOR(S):

ARAKAWA, TOMOYUKI

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INFINITE DIMENSIONAL LIE ALGEBRAS, VERTEX ALGEBRAS AND W-ALGEBRAS

TOMOYUKI ARAKAWA
DEPARTMENT OF MATHEMATICS, NARA WOMEN'S UNIVERSITY

1. INTRODUCTION

1.1. One of the distinguished features of infinite dimensional Lie algebras is the modular invariance of the characters of certain representations. There are two celebrated examples for this phenomena: One is the integrable highest weight representations of an affine Lie algebra $\widehat{\mathfrak{g}}$ associated with a simple Lie algebra \mathfrak{g} at a fixed level [KP], and the other is the minimal series representations [FFu] of the Virasoro algebra Vir with a fixed central charge.

However there is a relevant difference in these two examples: The Virasoro algebra is a single Lie algebra, while affine Lie algebras constitute a family of Lie algebras. Therefore it is natural to consider a generalization of the Virasoro algebra.

The *W-algebras* can be regarded as such a generalization of the Virasoro algebra. Some people say that this is the reason why they are called the “W-algebras” (because the letter “W” comes right after “V” alphabetically). The first example of a W-algebra was discovered by Zamalodchikov [Za] in his study of classification of conformal field theory (see [BS] and reference therein.).

1.2. In general, there is the W-algebra $\mathcal{W}(\mathfrak{g})$ associated with any simple Lie algebra \mathfrak{g} ([FF2]). The simplest W-algebra is the W-algebra $\mathcal{W}(\mathfrak{sl}_2)$ associated with \mathfrak{sl}_2 . This is nothing but the Virasoro algebra (or more precisely, the corresponding vertex algebra). The Virasoro algebra Vir is the Lie algebra with the following generators and the relations:

$$\begin{aligned} \text{generators: } & L_n \ (n \in \mathbb{Z}), \ c \\ \text{relations: } & [L_n, c] = 0 \\ & [L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m+n,0}c. \end{aligned}$$

The next simplest W-algebra is the one associated with \mathfrak{sl}_3 ; $\mathcal{W}(\mathfrak{sl}_3)$ is defined by the following generators and relations:

$$\begin{aligned} \text{generators: } & \mathbf{c}, L_n \ (n \in \mathbb{Z}), W_n \ (n \in \mathbb{Z}), \\ \text{relations: } & [\mathbf{c}, \mathcal{W}(\mathfrak{sl}_3)] = 0, \\ & [L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c}, \\ & [L_m, W_n] = (2m-n)W_{m+n}, \\ & [W_m, W_n] \\ & = (m-n)\left\{\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right\}L_{m+n} \\ & + \frac{16}{22+5\mathbf{c}}(m-n)\Lambda_{m+n} + \frac{\mathbf{c}}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}, \end{aligned}$$

where

$$(1) \quad \Lambda_n = \sum_{k \leq 0} L_k L_{n-k} + \sum_{k \geq 0} L_{n-k} L_k - \frac{3}{10}(n+2)(n+3)L_n.$$

In the above formula, the pole at $\mathbf{c} = -22/5$ can be removed if we multiply W_n by $22 + 5\mathbf{c}$, and therefore it is inessential. More serious is the existence of the infinite sum of the quadratic term of the form $L_{n-k}L_k$. This means that the above *does not define* a Lie algebra in the usual sense. In general, W-algebras are no more Lie algebras and one should understand them as *vertex algebras* (see [K2, FB, BD] for the definition of vertex algebras).

1.3. As we have seen in the above, $\mathcal{W}(\mathfrak{g})$ has a complicated algebraic structure except for the case that $\mathfrak{g} = \mathfrak{sl}_2$. In fact, even the defining relations of the generators are not known for a general $\mathcal{W}(\mathfrak{g})$! Thus, instead of defining it by generators and relations, W-algebras are usually defined by a cohomological method. This method is called the *quantized Drinfeld-Sokolov reduction*, or simply the *quantum reduction*, and was discovered by Feigin and Frenkel [FF2]. This is a powerful method, in the sense that it not only gives a uniform definition of $\mathcal{W}(\mathfrak{g})$, but also defines a functor from a suitable category (the category \mathcal{O}) of $\hat{\mathfrak{g}}$ -modules to the category of $\mathcal{W}(\mathfrak{g})$ -modules. Frenkel, Kac and Wakimoto [FKW] conjectured that one can obtain a family of modular invariant representations of $\mathcal{W}(\mathfrak{g})$ from the modular invariant representations (admissible representations) of $\hat{\mathfrak{g}}$ via this functor. If this is true then one can surely say that $\mathcal{W}(\mathfrak{g})$ is a generalization of *Vir*, for it inherits our favorite property of the Virasoro algebra.

1.4. The propose of this note to describe the representation theory of $\mathcal{W}(\mathfrak{g})$ via quantum reduction. In particular, we explain how the conjecture of Frenkel, Kac and Wakimoto follows from our general results.

2. FINITE DIMENSIONAL CASE

2.1. Recall that $\hat{\mathfrak{g}}$ is an affinization (or a *chiralization*) of the finite dimensional Lie algebra \mathfrak{g} . In this sense, the Virasoro algebra *Vir* is a chiralization of its zero mode, “ CL_0 ”. And because L_0 corresponds to the Casimir operator (via the Sugawara construction), one can think of $Vir = \mathcal{W}(\mathfrak{sl}_2)$ as a chiralization of the center $\mathcal{Z}(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$. This is true in general:

$\mathcal{W}(\mathfrak{g})$ is a chiralization of the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$.

2.2. Kostant's Theorem. Let e be a principal nilpotent element of \mathfrak{g} . For instance, if $\mathfrak{g} = \mathfrak{sl}_n$, then e has the form

$$e = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By the Jacobson-Morozov theorem there exists a corresponding \mathfrak{sl}_2 -triple $\{e, h_0, f\}$:

$$[h_0, e] = 2e, \quad [h_0, f] = -2f, \quad [e, f] = h_0.$$

Then we have the eigenspace decomposition of \mathfrak{g} with respect to the adjoint action of $\rho^\vee := h_0/2$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g}; [\rho^\vee, x] = jx\}.$$

Because e is principal, this gives a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where

$$\mathfrak{n}_+ = \sum_{j>0} \mathfrak{g}_j, \quad \mathfrak{h} = \mathfrak{g}_0, \quad \mathfrak{n}_- = \sum_{j<0} \mathfrak{g}_j.$$

Let $\Delta_+ \subset \mathfrak{h}^*$ be the corresponding set of positive roots, $\Delta_- = -\Delta_+$, $\Delta = \Delta_+ \sqcup \Delta_-$.

Define $p \in \mathfrak{n}_-^*$ by

$$p(x) = (x, e).$$

Here (\cdot, \cdot) is the normalized invariant inner product of \mathfrak{g} . Then $p([\mathfrak{n}_-, \mathfrak{n}_-]) = 0$ and p defines a character of \mathfrak{n}_- .

Let \mathcal{Cl} be the Clifford algebra associated with the space $\mathfrak{n}_- \oplus \mathfrak{n}_-^*$ and the natural bilinear form on it. Then \mathcal{Cl} has the following generators and relations:

generators: $\psi_\alpha, \psi_\alpha^* \quad (\alpha \in \Delta_-),$

relations: $\{\psi_\alpha, \psi_\beta^*\} = \delta_{\alpha, \beta}, \quad \{\psi_\alpha, \psi_\beta\} = \{\psi_\alpha^*, \psi_\beta^*\} = 0.$

We shall regard

$$U(\mathfrak{g}) \otimes \mathcal{Cl}$$

as a superalgebra with even generators $\mathfrak{g} \ni x = x \otimes 1$ and odd generators $\psi_\alpha = 1 \otimes \psi_\alpha, \psi_\alpha^* = 1 \otimes \psi_\alpha^*$.

Define an odd element $Q^{\text{st}} \in U(\mathfrak{g}) \otimes \mathcal{Cl}$ by

$$Q^{\text{st}} = \sum_{\alpha \in \Delta_-} x_\alpha \psi_\alpha^* - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_-} c_{\alpha, \beta}^\gamma \psi_\alpha^* \psi_\beta^* \psi_\gamma.$$

Here x_α is a (fixed) root vector of root α and $c_{\alpha, \beta}^\gamma$ is the structure constant. Then by direct calculation one can check that $[Q^{\text{st}}, Q^{\text{st}}] = 0$, or equivalently,

$$(Q^{\text{st}})^2 = 0.$$

We remark that the "st" suffix stands for "standard", because Q^{st} is the differential of the standard Lie algebra cohomology or homology.

Set

$$Q := Q^{\text{st}} + p,$$

where p is considered as an element of $Cl \subset U(\mathfrak{g}) \otimes Cl$:

$$p = \sum_{\alpha \in \Delta_-} p(x_\alpha) \psi_\alpha^*.$$

Lemma 1. $[p, p] = [Q^{\text{st}}, p] = 0$. Therefore $[Q, Q] = 0$, or equivalently $Q^2 = 0$.

By Lemma 1 it follows that

$$(\text{ad } Q)^2 = 0$$

on $U(\mathfrak{g}) \otimes Cl$. Hence we can consider $(U(\mathfrak{g}) \otimes Cl, \text{ad } Q)$ as a homology complex by setting

$$\begin{aligned} \deg u &= 0 \quad (u \in U(\mathfrak{g})), \\ \deg \psi_\alpha &= 1, \quad \deg \psi_\alpha^* = -1 \quad (\alpha \in \Delta_-). \end{aligned}$$

Then the corresponding homology

$$H_\bullet(U(\mathfrak{g}) \otimes Cl, \text{ad } Q) = \bigoplus_{i \in \mathbb{Z}} H_i(U(\mathfrak{g}) \otimes Cl, \text{ad } Q)$$

inherits the graded superalgebra structure from $U(\mathfrak{g}) \otimes Cl$.

Theorem 1 (Kostant [Ko], Kostant-Sternberg [KS], cf. [A3, Theorem 2.3.2]).

- (i) $H_{i \neq 0}(U(\mathfrak{g}) \otimes Cl, \text{ad } Q) = 0$.
- (ii) The map

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \rightarrow & H_0(U(\mathfrak{g}) \otimes Cl, \text{ad } Q) \\ z & \mapsto & z \otimes 1 \end{array}$$

is an isomorphism of \mathbb{C} -algebras.

2.3. Reduction Functor. Let $\Lambda(\mathfrak{n}_-)$ be the Grassmann algebra of \mathfrak{n}_- . Then $\Lambda(\mathfrak{n}_-)$ is naturally a module over Cl . Thus, for a \mathfrak{g} -module M ,

$$C(M) := M \otimes \Lambda(\mathfrak{n}_-)$$

is naturally a module over $U(\mathfrak{g}) \otimes Cl$. Thus, $(C(M), Q)$ again has the structure of homology complex. Let

$$H_\bullet(M) := H_\bullet(C(M), Q).$$

By definition $(C(M), Q)$ is identical to the Chevalley complex for calculating the Lie algebra homology $H_\bullet(\mathfrak{n}_-, M \otimes \mathbb{C}_p)$. Hence

$$(2) \quad H_\bullet(M) = H_\bullet(\mathfrak{n}_-, M \otimes \mathbb{C}_p).$$

On the other hand, the $U(\mathfrak{g}) \otimes Cl$ -module structure of $C(M)$ induces a $\mathcal{Z}(\mathfrak{g})$ -module structure on $H_i(M)$, because $\mathcal{Z}(\mathfrak{g}) = H_0(U(\mathfrak{g}) \otimes Cl, \text{ad } Q)$. Therefore we have obtained the following functor:

$$(3) \quad \begin{array}{ccc} H_i(?) : \mathfrak{g}\text{-Mod} & \rightarrow & \mathcal{Z}(\mathfrak{g})\text{-Mod} \\ M & \mapsto & H_i(M). \end{array}$$

Let \mathcal{O} be the BGG category [BGG] of \mathfrak{g} . Let $M(\lambda) \in \mathcal{O}$ the Verma module of highest weight λ , $L(\lambda) \in \mathcal{O}$ the unique irreducible quotient of $M(\lambda)$. Then it is known that the following are equivalent:

- (i) The Gelfand-Kirillov dimension $\text{Dim } L(\lambda)$ of $L(\lambda)$ is maximal, i.e. $\text{Dim}(L(\lambda)) = \dim \mathfrak{n}_-$.
- (ii) $L(\lambda) = M(\lambda)$,

(iii) λ is anti-dominant, i.e. $\lambda(\alpha^\vee) \notin \mathbb{N}$ for all $\alpha \in \Delta_+$.

The following assertion was essentially proved by Kostant [Ko] (cf. [A3, Section 2])

Theorem 2.

- (i) $H_{i \neq 0}(M) = 0$ for all $M \in \mathcal{O}$.
- (ii) $H_0(L(\lambda)) = \begin{cases} \mathbb{C}_{\gamma_\lambda} & \text{if } \dim L(\lambda) = \dim \mathfrak{n}_-, \\ 0 & \text{if } \dim L(\lambda) < \dim \mathfrak{n}_-. \end{cases}$

Here $\mathbb{C}_{\gamma_\lambda} = \mathcal{Z}(\mathfrak{g}) / \text{Ker } \gamma_\lambda$ and $\gamma_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is the central character defined as the evaluation at $M(\lambda)$.

By Theorem 2 (i), the functor $H_0(?)$ is exact. Moreover, by Theorem 2 (ii), one can obtain each simple $\mathcal{Z}(\mathfrak{g})$ -module as the image of the functor $H_0(?)$.

Remark 1. More is known for the functor $H_0(?)$. According to Soergel [S] and Backelin [Ba], it holds that

$$\text{Hom}_{\mathcal{O}}(M, P) \cong \text{Hom}_{\mathcal{Z}(\mathfrak{g})}(H_0(M), H_0(P))$$

provided that P is projective in \mathcal{O} (cf. [A3, Section 2]).

3. CHIRALIZATION OF THE CENTER

3.1. We now “chiralize” the construction of the previous section to define affine W-algebras. To this end we “chiralize” the every data used for the cohomological realization of $\mathcal{Z}(\mathfrak{g})$ in Theorem 2. Thus

- \mathfrak{g} is replaced by the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$, where K is the central element and D is the degree operator;
- \mathfrak{n}_- is replaced by its loop algebra $L\mathfrak{n}_- = \mathfrak{n}_- \otimes \mathbb{C}[t, t^{-1}] \subset \widehat{\mathfrak{g}}$;
- $\mathcal{C}l$ is replaced by the Clifford algebra $\widehat{\mathcal{C}l}$ associated with $L\mathfrak{n}_- \oplus (L\mathfrak{n}_-)^*$ and its natural symmetric bilinear form, where $(L\mathfrak{n}_-)^*$ is the graded dual of $L\mathfrak{n}_-$. This algebra may be defined by the following generators and relations:

$$\begin{aligned} \text{generators: } & \psi_\alpha(n), \psi_\alpha^*(n) \quad (\alpha \in \Delta_-, n \in \mathbb{Z}), \\ \text{relations: } & \{\psi_\alpha(m), \psi_\beta^*(n)\} = \delta_{\alpha, \beta} \delta_{m+n, 0}, \\ & \{\psi_\alpha(m), \psi_\beta(n)\} = \{\psi_\alpha^*(m), \psi_\beta^*(n)\} = 0; \end{aligned}$$

- $Q = Q^{\text{st}} + p$ is replaced by $\widehat{Q} = \widehat{Q}^{\text{st}} + \widehat{p}$, where

$$\begin{aligned} \widehat{Q}^{\text{st}} &= \sum_{\substack{\alpha \in \Delta_- \\ k \in \mathbb{Z}}} x_\alpha(-k) \psi_\alpha^*(k) - \frac{1}{2} \sum_{\substack{\alpha, \beta, \gamma \in \Delta_- \\ k+l+m=0}} c_{\alpha, \beta}^\gamma \psi_\alpha^*(k) \psi_\beta^*(l) \psi_\gamma(m), \\ \widehat{p} &= \sum_{\alpha \in \Delta_-} p(x_\alpha) \psi_\alpha^*(0), \end{aligned}$$

$$\text{where } x(k) = x \otimes t^k \in \widehat{\mathfrak{g}}.$$

By analogy with Theorem 1, we want to define the affine W-algebra $\mathcal{W}(\mathfrak{g})$ as

$$“\mathcal{W}(\mathfrak{g}) = H_0(U(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}, \text{ad } \widehat{Q})”.$$

But this does not make sense, for the appearance of the infinite sum in the formula of \widehat{Q}^{st} . Thus we need to make a suitable completion of $U(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}$. We also specialize

the value of the central element $K \in \widehat{\mathfrak{g}}$ at a given complex number $k \in \mathbb{C}$. So let $U_k(\widehat{\mathfrak{g}}) = U(\widehat{\mathfrak{g}})/(K - k \text{id})$. The algebra $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}$ is naturally graded:

$$U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l} = \bigoplus_{d \in \mathbb{Z}} (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l})_d,$$

where the grading is taken from the relation

$$(4) \quad \deg x(n) = \deg \psi_\alpha(n) = \deg \psi_\alpha^*(n) = n, \deg 1 = 0.$$

Give $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}$ the linear topology defined by the decreasing sequence where

$$\mathcal{I}_N = \bigoplus_{d \in \mathbb{Z}} (\mathcal{I}_N)_d, \quad (\mathcal{I}_N)_d = \sum_{j \geq N} (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l})_{d-j} (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l})_j.$$

Let $\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}$ be the corresponding completion:

$$\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}} = \varprojlim_N (U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l} / \mathcal{I}_N).$$

Then Q is a well-defined element of the topological algebra $\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}$, and one can define

$$(5) \quad H_\bullet(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}) := \varprojlim_N H_\bullet(U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l} / \mathcal{I}_N, \text{ad } \widehat{Q}).$$

But

$$(6) \quad \mathcal{W}_k(\mathfrak{g}) = H_0(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}) \quad (k \in \mathbb{C}).$$

is still not a correct definition of W-algebra, because what is defined by (6) is a topological algebra in the usual sense, but an affine W-algebra should be defined as a vertex algebra. So what we actually mean by (6) is the following statement:

Theorem 3 ([A3, Theorem 3.11.1]). *There is an isomorphism*

$$\mathcal{U}(\mathcal{W}_k(\mathfrak{g})) \cong H_0(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}),$$

where $\mathcal{U}(V) = \bigoplus_{d \in \mathbb{Z}} \mathcal{U}(V)_d$ is the universal enveloping algebra of a vertex algebra V (in the sense of Frenkel and Zhu [FZ]).

Remark 2. The vanishing $H_{i \neq 0}(\widetilde{U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{C}l}}, \text{ad } \widehat{Q}) = 0$ also holds.

We will not define the W-algebra $\mathcal{W}_k(\mathfrak{g})$ itself in this note. Instead, we take (6) as its definition because a $\mathcal{W}_k(\mathfrak{g})$ -module M is by definition a $\mathcal{U}(\mathcal{W}_k(\mathfrak{g}, e))$ -module (such that $\dim \mathcal{U}(\mathcal{W}_k(\mathfrak{g}, e))_n \cdot v < \infty$ for all $v \in V$ and $n \geq 0$). But it should be remarked that Theorem 3 follows from the corresponding statement for the vertex algebra $\mathcal{W}_k(\mathfrak{g})$ itself. This was proved for generic k by Feigin and Frenkel [FF2], for a general k and $\mathfrak{g} = \mathfrak{sl}_n$ by de Bore and Tjin [dBT2] and for a general k and a general \mathfrak{g} by Frenkel [FB].

Remark 3. The W-algebra $\mathcal{W}_k(\mathfrak{g})$ considered here is not a simple vertex algebra in general.

Remark 4. If $k \neq -h^\vee$, then $\mathcal{W}_k(\mathfrak{g})$ has the structure of the vertex operator algebra and has the central charge

$$c(k) = \text{rank } \mathfrak{g} - 12(\kappa |\rho^\vee|^2 - \langle \rho, \rho^\vee \rangle + |\rho|^2 / \kappa), \quad (\kappa = k + h^\vee).$$

Remark 5. It is known that $\mathcal{W}_{-k^\vee}(\mathfrak{g})$ is commutative. This is one of the results of Feigin-Frenkel [FF2].

To give a more precise relationship between $\mathcal{Z}(\mathfrak{g})$ and $\mathcal{W}_k(\mathfrak{g})$, let us introduce the notion of *Zhu algebra* $\text{Zh}(V)$ of a (graded) vertex algebra V .

$$\text{Zh}(V) := \mathcal{U}(V)_0 / \overline{\sum_{p>0} \mathcal{U}(V)_{-p} \mathcal{U}(V)_p},$$

where $\overline{}$ denotes the closure. By definition the following assertion is clear.

Theorem 4 (Zhu [Zhu]). *There is a one-to-one correspondence between irreducible V -modules and irreducible $\text{Zh}(V)$ -modules.*

For example, consider the universal affine vertex algebra $V_k(\mathfrak{g})$ associated with \mathfrak{g} at level k . Then $\mathcal{U}(V_k(\mathfrak{g})) = \widehat{U_k(\widehat{\mathfrak{g}})}$ and we have $\text{Zh}(V_k(\mathfrak{g})) = U(\mathfrak{g})$. This reflects the fact that $\widehat{\mathfrak{g}}$ (or more precisely $V_k(\mathfrak{g})$) is a chiralization of \mathfrak{g} . Since $\mathcal{W}_k(\mathfrak{g})$ is a chiralization of $\mathcal{Z}(\mathfrak{g})$, it is natural to expect the following assertion:

Theorem 5 ([A3, Theorem 3.13.2]). *The Zhu algebra $\text{Zh}(\mathcal{W}_k(\mathfrak{g}))$ of $\mathcal{W}_k(\mathfrak{g})$ is naturally isomorphic to $\mathcal{Z}(\mathfrak{g})$.*

By Theorems 4, 5, irreducible $\mathcal{W}_k(\mathfrak{g})$ -modules are parameterized by the central characters of $\mathcal{Z}(\mathfrak{g})$. Let $L(\gamma)$ denote the irreducible $\mathcal{W}_k(\mathfrak{g})$ -module corresponding to the central character γ . Then $L(\gamma)$ is the quotient of the Verma module $M(\gamma)$ with highest weight γ , which has the PBW type basis.

3.2. As in the finite dimensional case we functionally obtain the $\mathcal{W}_k(\mathfrak{g})$ -modules in the following way: Let $\Lambda^{\frac{\infty}{2}}(Ln_-)$ be the irreducible representation of $\widehat{\mathcal{Cl}}$ generated by the vector $\mathbf{1}$ satisfying the following relations:

$$\psi_\alpha(n)\mathbf{1} = \psi_\alpha^*(n+1)\mathbf{1} = 0 \quad (\alpha \in \Delta_-, n \geq 0).$$

Denote by $\widehat{\mathcal{O}}_k$ the BGG category of $\widehat{\mathfrak{g}}$ at level k . Then

$$\widehat{C}(M) := M \otimes \Lambda^{\frac{\infty}{2}}(Ln_-)$$

with $M \in \widehat{\mathcal{O}}_k$ is naturally a module over $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{Cl}}$, and its action extends to the smooth action of $U_k(\widehat{\mathfrak{g}}) \otimes \widehat{\mathcal{Cl}}$. In particular the action of \widehat{Q} is well-defined on $\widehat{C}(M)$. Thus the homology

$$(7) \quad \widehat{H}_\bullet(M) := H_\bullet(\widehat{C}(M), \widehat{Q})$$

well-defined and is naturally a module over $\mathcal{W}_k(\mathfrak{g})$. Note that $\widehat{H}_i(M)$ is naturally graded (cf. (4)):

$$(8) \quad \widehat{H}_i(M) = \bigoplus_{d \in \mathbb{C}} \widehat{H}_i(M)_d.$$

If k is not critical then (8) is essentially the L_0 -eigenspace decomposition. Set $\text{ch } \widehat{H}_i(M) = \sum_{d \in \mathbb{C}} q^d \dim H_i(M)_d$ whenever it is well-defined.

Remark 6. By definition we have $\widehat{H}_\bullet(M) = H_{\frac{\infty}{2}+\bullet}(Ln_-, M \otimes \mathbb{C}_{\widehat{p}})$, the Feigin's semi-infinite Ln_- -homology with the coefficient in $M \otimes \mathbb{C}_{\widehat{p}}$ ([Fe]).

Let $\widehat{M}(\widehat{\lambda})$ be the Verma module of $\widehat{\mathfrak{g}}$ with highest weight $\widehat{\lambda}$, $\widehat{L}(\widehat{\lambda})$ the unique simple quotient of $\widehat{M}(\widehat{\lambda})$.

Theorem 6 ([A3]). *For any $k \in \mathbb{C}$ we have the following.*

- (i) $\widehat{H}_{i \neq 0}(M) = 0$ for all objects M of $\widehat{\mathcal{O}}_k$.
- (ii) Let $\widehat{\lambda}$ be a weight of $\widehat{\mathfrak{g}}$ at level k , λ the classical part of $\widehat{\lambda}$ (i.e. the restriction of $\widehat{\lambda}$ to \mathfrak{h}). Then

$$\widehat{H}_0(\widehat{L}(\widehat{\lambda})) = \begin{cases} L(\gamma_\lambda) & \text{if } \dim L(\lambda) = \dim \mathfrak{n}_-, \\ 0 & \text{if } \dim L(\lambda) < \dim \mathfrak{n}_-. \end{cases}$$

By Theorem 6 it follows that the functor

$$\widehat{H}_0(?) : \mathcal{O}_k \rightarrow \mathcal{W}_k(\mathfrak{g})\text{-Mod}$$

is exact for any $k \in \mathbb{C}$.

Write the formal character $\text{ch } \widehat{L}(\widehat{\lambda})$ of $L(\lambda)$ as

$$\text{ch } \widehat{L}(\widehat{\lambda}) = \sum_{\widehat{\mu}} m_{\widehat{\lambda}, \widehat{\mu}} \text{ch } \widehat{M}(\widehat{\mu}), \quad (m_{\widehat{\lambda}, \widehat{\mu}} \in \mathbb{Z}).$$

Then the following assertion follows from Theorem 6:

Theorem 7 ([A3]). $\text{ch } \widehat{H}_0(\widehat{L}(\widehat{\lambda})) = \sum_{\widehat{\mu}} m_{\widehat{\lambda}, \widehat{\mu}} q^{\widehat{\mu}(\mathbf{D})} \prod_{i \geq 0} (1 - q^{-i})^{-\text{rank } \mathfrak{g}}$.

Recall that the integer $m_{\lambda, \mu}$ is known by Kashiwara-Tanisaki-Cassian [KT1, KT2, KT3, Ca] provided that $k \neq -h^\vee$. Therefore by Theorems 6 and 7 we have obtained the character formula of *all* the irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g})$ for any $k \in \mathbb{C} \setminus \{-h^\vee\}$.

Remark 7. It may be worth emphasizing that Theorems 6 and 7 remain valid even at the critical level $k = -h^\vee$, and the result for this case in particular implies the Kac-Kazhdan conjecture [KK], which was proved by Hayashi [Ha] and others [GW, FF1, Ku] by computational methods (see [A4] for details).

3.3. Frenkel-Kac-Wakimoto Conjecture. Note that our functor $\widehat{H}_0(?)$ kills integrable representations of $\widehat{\mathfrak{g}}$. However there are a wider class of modular invariant representations of $\widehat{\mathfrak{g}}$; they are called Kac-Wakimoto *admissible representations* [KW1, KW2].

The simple module $\widehat{L}(\widehat{\lambda})$ is called *admissible* if $\widehat{\lambda}$ is an *admissible weight*. An *admissible weight* is a weight $\widehat{\lambda}$ that satisfies the following:

- (i) $\widehat{\lambda}$ is regular dominant;
- (ii) the \mathbb{Q} -span of $\widehat{\Delta}(\widehat{\lambda})^\vee := \{\alpha \in \widehat{\Delta}_+^{\vee, \text{re}}; \widehat{\lambda}(\alpha) \in \mathbb{Z}\} = \text{the } \mathbb{Q}\text{-span of } \widehat{\Delta}_+^{\vee, \text{re}}.$

The condition (i) implies that the corresponding Kazhdan-Lusztig polynomial is trivial. Therefore $\widehat{L}(\widehat{\lambda})$ has the Weyl-Kac type character formula:

$$\text{ch } \widehat{L}(\widehat{\lambda}) = \sum_{w \in \widehat{W}(\widehat{\lambda})} (-1)^{\ell(w)} \text{ch } \widehat{M}(w \circ \widehat{\lambda}),$$

where $\widehat{W}(\widehat{\lambda})$ is the integral Weyl group of $\widehat{\mathfrak{g}}$, generated by the reflections r_α with $\alpha^\vee \in \widehat{\Delta}(\widehat{\lambda})^\vee$. The condition (ii) implies that $\widehat{W}(\widehat{\lambda})$ is an infinite Coxeter group, and $\text{ch } \widehat{L}(\widehat{\lambda})$ is written in terms of some theta functions ([KW1, KW2]).

If the classical part λ of an admissible weight $\widehat{\lambda}$ is anti-dominant, then $\widehat{\lambda}$ is called a *non-degenerate admissible weight*. Let $Pr_k^{\text{non-deg}}$ be the set of non-degenerate admissible weight at level k .

And as explained in Introduction, the conjecture of Frenkel, Kac and Wakimoto [FKW] follows from Theorems 6 and 7:

Corollary 1 (Frenkel-Kac-Wakimoto Conjecture [FKW]). *Let $\widehat{\lambda}$ be an non-degenerated admissible weight of $\widehat{\mathfrak{g}}$, λ the classical part of $\widehat{\lambda}$. Then*

$$\text{ch } \mathbf{L}(\gamma_\lambda) = \sum_{w \in \widehat{W}(\widehat{\lambda})} (-1)^{\ell(w)} q^{(w \circ \widehat{\lambda})(D)} \prod_{i \geq 1} (1 - q^{-i})^{-\text{rank } \mathfrak{g}}.$$

As explained in [FKW], from Corollary 1 it follows that the (modified) characters of

$$\{\mathbf{L}(\gamma_\lambda); \lambda \text{ is the classical part of } \widehat{\lambda} \in Pr_k^{\text{non-deg}}\}$$

are modular invariant, i.e. the linear space spanned by their (modified) characters are invariant under the natural action of $SL_2(\mathbb{Z})$. In the case that $\mathfrak{g} = \mathfrak{sl}_2$, they are exactly the minimal series representations of *Vir*.

4. GENERALIZATION TO OTHER NILPOTENT ORBITS

4.1. In the above construction we started with the principal nilpotent element of \mathfrak{g} . However the above construction can be generalized to cases of other nilpotent elements:

Let e be a nilpotent element which corresponds to a nice parabolic subalgebra ([BW]) of \mathfrak{g} . Then it is straightforward to generalize the previous construction to e (cf. [dBT1, dBT2, KRW]). As a result, instead of $\mathcal{Z}(\mathfrak{g})$, we obtain the *finite W-algebra* $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ [dBT1] associated with (\mathfrak{g}, e) , which is the endomorphism ring of the *generalized Gelfand-Graev representation* ([Ka], cf. [Pr, GG, BG]). The corresponding affine W-algebra $\mathcal{W}_k(\mathfrak{g}, e)$ has $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ as its Zhu algebra (cf. [DK]).

We have the similar result as Theorems 6 and 7 for this case ([A5]); The difficulty is that the representation theory of $\mathcal{W}^{\text{fin}}(\mathfrak{g}, e)$ is not known very much in general, except for the type *A* cases; Recently Brundan and Kleshchev [BK] established important results on the representation theory of finite W-algebras for these cases. Thanks to their result, for the type *A* cases one obtains the character formula for each irreducible highest weight representations of $\mathcal{W}_k(\mathfrak{g}, e)$ (see [A5] for details).

If e does not corresponds to a nice parabolic subalgebra then the construction of W-algebras becomes more involving. The most general construction was made by Kac, Roan and Wakimoto [KRW], which applies to the Lie superalgebra case also. One of the remarkable discoveries of Kac, Roan and Wakimoto [KRW] is that almost all the superconformal algebras (such as $N = 2, 3, 4$ superconformal algebra) appears as a W-algebra associated with some Lie superalgebra \mathfrak{g} and its *minimal nilpotent element*. As principal nilpotent element cases, their representation theory (such as characters of irreducible representations) can be completely described through the reduction functor (see [A2] for details).

REFERENCES

- [A1] Arakawa, T.; *Vanishing of cohomology associated to quantized Drinfeld-Sokolov reduction*, Int. Math. Res. Notices, 2004, No. 15, 729-767.
- [A2] Arakawa, T.; *Representation Theory of Superconformal Algebras and the Kac-Roan-Wakimoto Conjecture*, Duke. Math. J. Vol. 130 (2005), No. 3, 435-478.
- [A3] Arakawa, T.; *Representation Theory of W-Algebras*, math.QA/0506056.
- [A4] Arakawa, T.; *A new proof of the Kac-Kazhdan conjecture*, Int. Math. Res. Notices Volume 2006, Article ID 27091, Pages 1-5.

- [A5] Arakawa, T; in preparation.
- [Ba] Backelin, E.; Representation of the category \mathcal{O} in Whittaker categories, Int. Math. Res. Notices 1997, no. 4, 153–172.
- [BW] Baur, Karin., Wallach, Nolan; *Nice parabolic subalgebras of reductive Lie algebras*, Represent. Theory 9 (2005), 1–29 (electronic).
- [BD] Beilinson, Alexander; Drinfeld, Vladimir; *Chiral algebras*, American Mathematical Society Colloquium Publications, 51. American Mathematical Society, 2004.
- [BGG] Bernstein, I. N., Gelfand, I. M., Gelfand, S. I.; *A certain category of g -modules*, Funkcional. Anal. Appl. 10 (1976), no. 2, 1–8.
- [BG] Brundan, J., Goodwin, S.; *Good grading polytopes*, preprint math.QA/0510205.
- [BS] *\mathcal{W} -symmetry*. Edited by P. Bouwknegt and K. Schoutens. Advanced Series in Mathematical Physics, 22. World Scientific Publishing Co., Inc., River Edge, NJ, 1995
- [BK] Brundan, J., Kleshchev, A.; *Representations of shifted Yangians*, math.RT/0508003.
- [dBT1] de Boer, J., Tjin, T.; *Quantization and representation theory of finite \mathcal{W} -algebras*, Comm. Math. Phys. 158 (1993), no. 3, 485–516.
- [dBT2] de Boer, J., Tjin, T.; *The relation between quantum \mathcal{W} -algebras and Lie algebras*, Comm. Math. Phys. 160(1997), 317–332.
- [Ca] Casian, L; *Proof of the Kazhdan-Lusztig conjecture for Kac-Moody algebras (the characters $\text{ch } L_{w\rho-\rho}$)*, Adv. Math., 119 (1996) 207–281.
- [DK] De Sole, A., Kac, V. ; *Finite vs affine \mathcal{W} -algebras*, Japan J. Math., Vol. 1, Issue 1 (2006) 137–261.
- [DS] Drinfeld, V. G., Sokolov, V. V.; *Lie algebra and the KdV type equations*, Soviet J. Math. 30, 1975–2036 (1985).
- [EK] Elashvili, A.G., Kac, V.G.; *Classification of Good Gradings of Simple Lie Algebras*, Amer. Math. Soc. Transl. (2) vol 213 (2005), 85–104.
- [Fe] Feigin, B. L.; *Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras*, Uspekhi Mat. Nauk 39 (1984), no. 2(236), 195–196.
- [FF1] Feigin, B., Frenkel, E.; *A family of representations of affine Lie algebras*, (Russian) Uspekhi Mat. Nauk 43 (1988), no. 5(263), 227–228; translation in Russian Math. Surveys 43 (1988), no. 5, 221–222.
- [FF2] Feigin, B., Frenkel, E.; *Quantization of the Drinfeld-Sokolov reduction*, Phys. Lett. B 246 (1990), no. 1–2, 75–81.
- [FFu] Feigin, B. L., Fuchs, D. B.; *Verma modules over the Virasoro algebra*, Topology (Leningrad, 1982), 230–245, Lecture Notes in Math., 1060, Springer, Berlin, 1984.
- [FB] Frenkel, E., Ben-Zvi, D.; *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, 88, 2001
- [FKW] Frenkel, E., Kac, V., Wakimoto, M.; *Characters and fusion rules for \mathcal{W} -algebras via quantized Drinfeld-Sokolov reduction*, Comm. Math. Phys. 147 (1992), no. 2, 295–328.
- [FZ] Frenkel, I. B., Zhu, Y.; *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. 66 (1992), no. 1, 123–168.
- [GG] Gan, W. L., Ginzburg, V.; *Quantization of Slodowy slices*,
- [GW] Goodman, R., Wallach, N.R.; *Higher-order Sugawara operators for affine Lie algebras*, Tras. Amer. Math. Soc. 315(1989), np. 1.
- [Ha] Hayashi, Takahiro; *Sugawara operators and Kac-Kazhdan conjecture*, Invent. Math. 94(1988), no. 1, 13–52.
- [K1] Kac, Victor G.; *Infinite-dimensional Lie algebras*, Cambridge University Press, 1985.
- [K2] Kac, Victor G.; *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, 1997.
- [KK] Kac, V. G., Kazhdan, D. A.; *Structure of representations with highest weight of infinite-dimensional Lie algebras*, Adv. in Math. 34 (1979), no. 1, 97–08.
- [KP] Kac, Victor G., Peterson, Dale H.; *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. 53 (1984), no. 2, 125–264.
- [KRW] Kac, V. G., Roan, S.-S., Wakimoto, M.; *Quantum reduction for affine superalgebras*, Commun. Math. Phys. 241 (2003), 307–342.
- [Ku] Ku, Jong Min; *Structure of the Verma module $M(-\rho)$ over Euclidean Lie algebras*, J. Algebra 124 (1989), no. 2, 367–387.
- [KW1] Kac, V. G., Wakimoto, M.; *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Nat. Acad. Sci. U.S.A. 85 (1988), no. 14, 4956–4960.

- [KW2] Kac, V. G., Wakimoto, M.; *Classification of modular invariant representations of affine algebras*, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 138–177, Adv. Set. Math. Phys., 7, World Sci. Publishing, Teaneck, NJ, 1989.
- [KW3] Kac, V. G., Wakimoto, M.; *Quantum Reduction and Representation Theory of Superconformal Algebras*, Adv. Math. 185 (2004) 400–458, *Corrigendum to: "Quantum reduction and representation theory of superconformal algebras"*, Adv. Math. 185 (2004), no. 2, 400–458; MR2060475, Adv. Math. 193 (2005), no. 2, 453–455.
- [KT1] Kashiwara, M., Tanisaki, T.; *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level II : Nonintegral case*, Duke Math. J., 84 (1996) 771–813.
- [KT2] Kashiwara, M., Tanisaki, T.; *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III. Positive rational case*, Mikio Sato; a great Japanese mathematician of the twentieth century. Asian J. Math. 2 (1998), no. 4, 779–832.
- [KT3] Kashiwara, M., Tanisaki, T. *Characters of irreducible modules with non-critical highest weights over affine Lie algebras. Representations and quantizations*, 275–296, China High. Educ. Press, Beijing, 2000.
- [Ka] Kawanaka, N.; *Generalized Gelfand-Graev representations and Ennola duality*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), 175–206, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.
- [Ko] Kostant, B.; *On Whittaker vectors and representation theory*, Invent. Math. 48 (1978), no. 2, 101–184. 329–387.
- [KS] Kostant, B., Sternberg, S.; *Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras*, Ann. Physics 176 (1987), no. 1, 49–113.
- [Pr] Premet, A.; *Special transverse slices and their enveloping algebras*, Adv. Math. 170 (2002), no. 1, 1–55.
- [S] Soergel, W.; *Kategorie \mathcal{O} , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*, J. Amer. Math. Soc. 3 (1990), no. 2, 421–445.
- [Za] Zamolodchikov, A. B.; *Infinite extra symmetries in two-dimensional conformal quantum field theory*, Teoret. Mat. Fiz. 65 (1985), no. 3, 347–359.
- [Zhu] Zhu, Y.; *Modular invariance of characters of vertex operator algebras*, Journal AMS 9(1996), 237–302.

E-mail address: arakawa@cc.nara-wu.ac.jp